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Identities on Two-Dimensional Algebras

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Abstract—In the paper we provide some polynomial identities for finite-dimensional algebras. A list of well known single polynomial identities is exposed and the classification of all two-dimensional algebras with respect to these identities is given.

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1. INTRODUCTION

It is known that many important algebras in use are so called PI-algebras, that is, algebras satisfying a certain set of polynomial identities. Therefore, the classification of such algebras, up to isomorphism, is of a great interest. Earlier we have given classification results for some important classes of two-dimensional PI-algebras [2–5, 7]. Subalgebras, idempotents, ideals and quasi-units of two- dimensional algebras are described in [1]. In this paper we consider a list of some important polynomial identities which have appeared earlier in the theory of algebras and present a classification of two-dimensional algebras with respect to these identities. For other results related to the classification problem and the problems raised in this paper we refer the reader to [8–14].

The organization of the paper is as follows. In the next section we introduce definitions, notations and results needed in the course of the study followed by two section where we present main results of the paper. In Section 3 we provide some polynomial identities for finite-dimensional algebras. The last section is devoted to the classification of two-dimensional algebras with respect to the identities specified.

2. PRELIMINARIES

In this paper an algebra $(\mathbb{A}, \cdot)$ means a vector space $\mathbb{A}$ over a field $\mathbb{F}$ with a given bilinear map $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$ and often we drop $\cdot$ in the writing.

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If $\mathbb{A}$ such a two-dimensional algebra and $\mathrm{e} = (e_1, e_2)$ is a fixed basis then by $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$ we denote its matrix of structure constants (MSC) with respect to this basis, i.e.,

$$e_1e_1 = \alpha_1 e_1 + \beta_1 e_2, \quad e_1e_2 = \alpha_2 e_1 + \beta_2 e_2, \quad e_2e_1 = \alpha_3 e_1 + \beta_3 e_2, \quad e_2e_2 = \alpha_4 e_1 + \beta_4 e_2.$$

Further it is assumed that the basis e is fixed and we do not make difference between an algebra $\mathbb{A}$ and its MSC A with respect to this basis.

The classification problem of all two-dimensional algebras over any field $\mathbb{F}$, where any second and third degree polynomial possess a root, has been solved in [6]. The result was given there by providing the canonical MSCs for such algebras. In this paper we rely on the result of [6], follow its notations and for a convenience we present below the corresponding canonical representatives of the isomorphism classes of all two-dimensional algebras in form of tables according to $\mathrm{Char}(\mathbb{F}) \neq 2, 3$, $\mathrm{Char}(\mathbb{F}) = 2$ and $\mathrm{Char}(\mathbb{F}) = 3$ cases. Note that the parameters given in the canonical representatives may take any values in $\mathbb{F}$.

	Algebra	Structure constants							
		α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
$\mathrm{Char}(\mathbb{F}) \neq 2, 3$	$A_1(\mathbf{c})$	α_1	α_2	α_2+1	α_4	β_1	$-\alpha_1$	$1-\alpha_1$	$-\alpha_2$
	$A_2(\mathbf{c})$	α_1	0	0	1	β_1	β_2	$1-\alpha_1$	0
	$A_3(\mathbf{c})$	0	1	1	0	β_1	β_2	1	-1
	$A_4(\mathbf{c})$	α_1	0	0	0	0	β_2	$1-\alpha_1$	0
	$A_5(\mathbf{c})$	α_1	0	0	0	1	$2\alpha_1-1$	$1-\alpha_1$	0
	$A_6(\mathbf{c})$	α_1	0	0	1	β_1	$1-\alpha_1$	$-\alpha_1$	0
	$A_7(\mathbf{c})$	0	1	1	0	β_1	1	0	-1
	$A_8(\mathbf{c})$	α_1	0	0	0	0	$1-\alpha_1$	$-\alpha_1$	0
	A_9	$\frac{1}{3}$	0	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0
	A_{10}	0	1	1	0	0	0	0	-1
	A_{11}	0	1	1	0	1	0	0	-1
	A_{12}	0	0	0	0	1	0	0	0

$$A_2(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{F}^3,$$

$$A_6(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{F}^2,$$

	Algebra	The structure constants							
		α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
Char $(\mathbb{F})=2$	$A_{1,2}(\mathbf{c})$	α_1	α_2	$1+\alpha_2$	α_4	β_1	α_1	$1+\alpha_1$	α_2
	$A_{2,2}(\mathbf{c})$	α_1	0	0	1	β_1	β_2	$1+\alpha_1$	0
	$A_{3,2}(\mathbf{c})$	α_1	1	1	0	0	β_2	$1+\alpha_1$	1
	$A_{4,2}(\mathbf{c})$	α_1	0	0	0	0	β_2	$1+\alpha_1$	0
	$A_{5,2}(\mathbf{c})$	α_1	0	0	0	1	1	$1+\alpha_1$	0
	$A_{6,2}(\mathbf{c})$	α_1	0	0	1	β_1	$1+\alpha_1$	α_1	0
	$A_{7,2}(\mathbf{c})$	α_1	1	1	0	0	$1+\alpha_1$	α_1	1
	$A_{8,2}(\mathbf{c})$	α_1	0	0	0	0	$1+\alpha_1$	α_1	0
	$A_{9,2}$	1	0	0	0	1	0	1	0
	$A_{10,2}$	0	1	1	0	0	0	0	1
	$A_{11,2}$	1	1	1	0	0	1	1	1
	$A_{12,2}$	0	0	0	0	1	0	0	0

	Algebra	The structure constants							
		α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
Char $(\mathbb{F})=3$	$A_{1,3}(\mathbf{c})$	α_1	α_2	α_2+1	α_4	β_1	$-\alpha_1$	$1-\alpha_1$	$-\alpha_2$
	$A_{2,3}(\mathbf{c})$	α_1	0	0	1	β_1	β_2	$1-\alpha_1$	0
	$A_{3,3}(\mathbf{c})$	0	1	1	0	β_1	β_2	1	-1
	$A_{4,3}(\mathbf{c})$	α_1	0	0	0	0	β_2	$1-\alpha_1$	0
	$A_{5,3}(\mathbf{c})$	α_1	0	0	0	1	$-\alpha_1-1$	$1-\alpha_1$	0
	$A_{6,3}(\mathbf{c})$	α_1	0	0	1	β_1	$1-\alpha_1$	$-\alpha_1$	0
	$A_{7,3}(\mathbf{c})$	0	1	1	0	β_1	1	0	-1
	$A_{8,3}(\mathbf{c})$	α_1	0	0	0	0	$1-\alpha_1$	$-\alpha_1$	0
	$A_{9,3}$	0	1	1	0	1	0	0	-1
	$A_{10,3}$	0	1	1	0	0	0	0	-1
	$A_{11,3}$	1	0	0	0	1	-1	-1	0
	$A_{12,3}$	0	0	0	0	1	0	0	0

$$A_{2,3}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{F}^3,$$

$$A_{6,3}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ -\beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{F}^2.$$

3. POLYNOMIAL IDENTITIES OF TWO-DIMENSIONAL ALGEBRAS

Let $\mathbb{F}$ be a field, $m \leq n \leq l$ be fixed natural numbers, S_n be the symmetric group and $x^1, x^2, ..., x^l$ be non-commutative, non-associative variables. By the use of these variables and parenthesis (,) one can construct different non-associative monomials (words) containing each of the variables $x^1, x^2, ..., x^l$ only once and they may occur in the monomial in any order. For example, in the case of $l = 2$ one has only two such monomials x^1x^2 and x^2x^1, whereas in the case of $l = 3$ there are the following twelve possibilities $\{(x^{\sigma(1)}x^{\sigma(2)})x^{\sigma(3)} : \sigma \in S_3\} \cup \{x^{\sigma(1)}(x^{\sigma(2)}x^{\sigma(3)}) : \sigma \in S_3\}$. Further $w(x^1, x^2, ..., x^l)$ stands for such a monomial and we associate with $w(x^1, x^2, ..., x^l)$ the following multi-linear polynomial

$$w_{n,d}(x^1, x^2, ..., x^l) = \sum_{\sigma \in S_n} sgn(\sigma) w(x^{\sigma(1)}, x^{\sigma(2)}, ..., x^{\sigma(n)}, x^{n+1}, x^{n+2}, ..., x^l).$$

Let $\mathbb{A}$ be any m-dimensional algebra over $\mathbb{F}$. We use the notations $[\mathbf{u}, \mathbf{v}] = \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u}$ and $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} - \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ for the commutator and the associator of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively. It is assumed that a basis $\mathrm{e} = \{e_j\}_{j=1,2,3,...,m}$ of $\mathbb{A}$ over $\mathbb{F}$ is fixed and $\mathbf{u}^i = \sum\limits_{j=1}^{m} x_j^i e_j \in \mathbb{A}$, where $i = 1, 2, , 3, ..., l$, stand for any elements of $\mathbb{A}$. Define

$$|\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^m| = \begin{vmatrix} x_1^1 & x_2^1 & \ldots & x_m^1 \\ x_1^2 & x_2^2 & \ldots & x_m^2 \\ \vdots & \vdots & \ldots & \vdots \\ x_1^m & x_2^m & \ldots & x_m^m \end{vmatrix}.$$

Theorem 3.1. *If $m = n = l$ then there exists such an element $\mathbf{u}_0 \in \mathbb{A}$ that for any $\mathbf{u}^i = \sum\limits_{j=1}^{m} x_j^i e_j \in \mathbb{A}$, where $i = 1, 2, , 3, ..., m$, the equality*

$$w_{m,d}(\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^m) = |\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^m| \mathbf{u}_0 \tag{1}$$

holds true. If $n > m$ then $w_d(\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^l) = 0$.

Proof. It is clear that $w(\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^l) = \sum\limits_{\substack{i_j=1,_{j=1,...,l,} \\ s=1}}^{m} x_{i_1}^1 x_{i_2}^2 ... x_{i_l}^l c_s^{(i_1,i_2,...,i_l)} e_s$, where $c_s^{(i_1,i_2,...,i_l)} \in \mathbb{F}$ and therefore,

$$w_{n,d}(\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^l) = \sum_{\substack{i_j=1,_{j=1,...,l,} \\ s=1}}^{m} \sum_{\sigma \in S_n} sgn(\sigma) x_{i_1}^{\sigma(1)} x_{i_2}^{\sigma(2)} ... x_{i_n}^{\sigma(n)} x_{i_{n+1}}^{n+1} x_{i_{n+2}}^{n+2} ... x_{i_l}^l c_s^{(i_1,i_2,...,i_l)} e_s$$

$$= \sum_{\substack{i_j=1,_{j=1,...,l,} \\ s=1}}^{m} \begin{vmatrix} x_{i_1}^1 & x_{i_2}^1 & ... & x_{i_n}^1 \\ x_{i_1}^2 & x_{i_2}^2 & ... & x_{i_n}^2 \\ \vdots & \vdots & ... & \vdots \\ x_{i_1}^n & x_{i_2}^n & ... & x_{i_n}^n \end{vmatrix} x_{i_{n+1}}^{n+1} x_{i_{n+2}}^{n+2} ... x_{i_l}^l c_s^{(i_1,i_2,...,i_l)} e_s.$$

The determinant in the expression above is zero whenever either two of the numbers $i_1, i_2, ..., i_n$ are the same and hence, in this case we have $w_{n,d}(\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^l) = 0$ whenever $n > m$, whereas if $m = n = l$ one has

$$w_{m,d}(\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^m) = |\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^m| \mathbf{u}_0, \quad \text{where} \quad \mathbf{u}_0 = \sum_{s=1}^{m} \sum_{\sigma \in S_m} c_s^{(\sigma(1),\sigma(2),...,\sigma(m))} e_s.$$

□

Corollary 3.2. *For any m-dimensional algebra $\mathbb{A}$ the following identity*

$$[w_{m,d}(\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^m), w_{m,d}(\mathbf{v}^1, \mathbf{v}^2, ..., \mathbf{v}^m)] = 0$$

holds true.

Further we deal only with two-dimensional algebras. Due to the theorem above in any two-dimensional algebra $\mathbb{A}$ the following identities hold true

$$[[\mathbf{u}, \mathbf{v}], [\mathbf{u}', \mathbf{v}']] = 0, \quad [\mathbf{u}, \mathbf{v}]\mathbf{w} + [\mathbf{v}, \mathbf{w}]\mathbf{u} + [\mathbf{w}, \mathbf{u}]\mathbf{v} = 0, \quad \mathbf{w}[\mathbf{u}, \mathbf{v}] + \mathbf{u}[\mathbf{v}, \mathbf{w}] + \mathbf{v}[\mathbf{w}, \mathbf{u}] = 0 \quad (2)$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u}', \mathbf{v}' \in \mathbb{A}$. In particular, for any such algebra $(\mathbb{A}, \cdot)$ the corresponding $(\mathbb{A}, [\cdot, \cdot])$ is a Lie algebra. Note that any three-linear identity in a two-dimensional algebra is a linear combination of the identities (2).

Let us now observe a few simple identities for some classes of two-dimensional algebras presented above in terms of their MSC (note that these identities also hold true for the corresponding isomorphic algebras). In the cases of A_4, A_5, $A_8(\alpha_1)$ and A_9 one has

$$[\mathbf{u}, \mathbf{v}] = |\mathbf{u}, \mathbf{v}|(\beta_2 + \alpha_1 - 1)e_2, \quad [\mathbf{u}, \mathbf{v}] = (3\alpha_1 - 2)|\mathbf{u}, \mathbf{v}|e_2, \quad [\mathbf{u}, \mathbf{v}] = (x_1y_2 - x_2y_1)e_2$$

and $[\mathbf{u}, \mathbf{v}] = (x_1y_2 - x_2y_1)e_2$, respectively. Due to $e_2^2 = 0$ these imply the identities $[\mathbf{u}, \mathbf{v}][\mathbf{u}', \mathbf{v}'] = 0$ and $[\mathbf{u}, \mathbf{v}, \mathbf{w}][\mathbf{u}', \mathbf{v}', \mathbf{w}'] = 0$. Moreover, for A_9 the identity $2[\mathbf{u}, \mathbf{v}]\mathbf{w} + \mathbf{w}[\mathbf{u}, \mathbf{v}] = 0$ also holds true. Indeed,

$$\mathbf{u}\mathbf{v} = (x_1e_1 + x_2e_2)(y_1e_1 + y_2e_2) = \frac{x_1y_1}{3}e_1 + \frac{3x_1y_1 + 2x_1y_2 - x_2y_1}{3}e_2,$$

$$[\mathbf{u}, \mathbf{v}] = (x_1y_2 - x_2y_1)e_2, \quad [\mathbf{u}, \mathbf{v}]\mathbf{w} = -\frac{z_1}{3}(x_1y_2 - x_2y_1)e_2, \quad \mathbf{w}[\mathbf{u}, \mathbf{v}] = \frac{2z_1}{3}(x_1y_2 - x_2y_1)e_2.$$

Therefore, we have $2[\mathbf{u}, \mathbf{v}]\mathbf{w} + \mathbf{w}[\mathbf{u}, \mathbf{v}] = 0$.

For algebras A_{10}, A_{11} and A_{12} one has $\mathbf{u}\mathbf{v} = \mathbf{v}\mathbf{u}$. Moreover, in the case of A_{10} we have

$$\mathbf{u}\mathbf{v} = (x_1e_1 + x_2e_2)(y_1e_1 + y_2e_2) = (x_1y_2 + x_2y_1)e_1 - x_2y_2e_2,$$

$$(\mathbf{u}\mathbf{v})\mathbf{w} = ((x_1y_2 + x_2y_1)z_2 - x_2y_2z_1)e_1 + x_2y_2z_2e_2,$$

$$\mathbf{u}(\mathbf{v}\mathbf{w}) = (-x_1y_2z_2 + x_2(y_1z_2 + y_2z_1))e_1 + x_2y_2z_2e_2,$$

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u}\mathbf{v})\mathbf{w} - \mathbf{u}(\mathbf{v}\mathbf{w}) = 2y_2(x_1z_2 - x_2z_1)e_1$$

and hence, the identities $[\mathbf{u}, \mathbf{v}, \mathbf{w}][\mathbf{u}', \mathbf{v}', \mathbf{w}'] = 0$, $[\mathbf{u}, \mathbf{v}, \mathbf{w}] + [\mathbf{v}, \mathbf{w}, \mathbf{u}] - [\mathbf{w}, \mathbf{u}, \mathbf{v}] = 0$ hold true.

In A_{11} we have

$$\mathbf{u}\mathbf{v} = (x_1e_1 + x_2e_2)(y_1e_1 + y_2e_2) = (x_1y_2 + x_2y_1)e_1 + (x_1y_1 - x_2y_2)e_2,$$

$$(\mathbf{u}\mathbf{v})\mathbf{w} = ((x_1y_2 + x_2y_1)z_2 + (x_1y_1 - x_2y_2)z_1)e_1 + ((x_1y_2 + x_2y_1)z_1 - (x_1y_1 - x_2y_2)z_2)e_2,$$

$$\mathbf{u}(\mathbf{v}\mathbf{w}) = (x_1(y_1z_1 - y_2z_2)z_2 + x_2(y_1z_2 + y_2z_1))e_1 + (x_1(y_1z_2 + y_2z_1) - x_2(y_1z_1 - y_2z_2))e_2,$$

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u}\mathbf{v})\mathbf{w} - \mathbf{u}(\mathbf{v}\mathbf{w}) = 2(x_1z_2 - x_2z_1)(y_2e_1 - y_1e_2)$$

and therefore, $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}, \mathbf{u}]$, $[\mathbf{u}, \mathbf{v}, \mathbf{w}] + [\mathbf{v}, \mathbf{w}, \mathbf{u}] + [\mathbf{w}, \mathbf{u}, \mathbf{v}] = 0$ hold true for the algebra A_{11}.

In the case of A_{12} the identities $(\mathbf{u}\mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v}\mathbf{w}) = 0$ are valid.

4. CLASSIFICATION OF TWO-DIMENSIONAL ALGEBRAS WITH RESPECT TO IDENTITIES

Recall that in the opposite algebra $(\mathbb{A}^{op}, *)$ to an algebra $(\mathbb{A}, \cdot)$ the product $*$ is defined by $\mathbf{u} * \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. Observe that an algebra $(\mathbb{A}, \cdot)$ satisfies a polynomial identity if and only if its "opposite" $(\mathbb{A}^{op}, *)$ satisfies the corresponding "opposite" polynomial identity, i.e., the polynomial identities of an "opposite" algebra can be derived from the polynomial identities of the original algebra.

For an algebra $\mathbb{A}$ with MSC A the MSC A^{op} of $\mathbb{A}^{op}$ can be easily found. In two-dimensional case it is evident that MSC of $\mathbb{A}^{op}$ with respect to the basis e is

$$A^{op} = \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_2 & \alpha_4 \\ \beta_1 & \beta_3 & \beta_2 & \beta_4 \end{pmatrix}, \quad \text{provided that} \quad A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

is MSC of $\mathbb{A}$ with respect to e.

The main motivation to raise the question on the opposite algebras is the fact that the list of canonical representatives presented in Section 2 is not invariant with respect to the "opposite" operation and therefore to study their behavior with respect to this operation is important, moreover, later on we use the result in the classification of two-dimensional algebras with respect to identities. Here are the results on "opposite"s of two-dimensional algebras.

Theorem 4.1. *Let Char*$(\mathbb{F}) \neq 2, 3$. *The following hold true:*

- $(A_1(\alpha_1, \alpha_2, \alpha_4, \beta_1))^{op} \cong A_1(-\alpha_2, -\alpha_1, \beta_1, \alpha_4)$,
- $(A_2(\alpha_1, \beta_1, \beta_2))^{op} \cong A_2\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{\beta_1}{(\alpha_1+\beta_2)\sqrt{\alpha_1+\beta_2}}, \frac{1-\alpha_1}{\alpha_1+\beta_2}\right)$, *whenever* $\alpha_1 + \beta_2 \neq 0$, $(A_2(\alpha_1, \beta_1, -\alpha_1))^{op} = A_6(\alpha_1, \beta_1)$,
- $(A_3(\beta_1, \beta_2))^{op} \cong A_3\left(\frac{\beta_1}{\beta_2^2}, \frac{1}{\beta_2}\right)$, *whenever* $\beta_2 \neq 0$, $(A_3(\beta_1, 0))^{op} = A_7(\beta_1)$,
- $(A_4(\alpha_1, \beta_2))^{op} \cong A_4\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{1-\alpha_1}{\alpha_1+\beta_2}\right)$, *whenever* $\alpha_1 + \beta_2 \neq 0$, $(A_4(\alpha_1, -\alpha_1))^{op} = A_8(\alpha_1)$,
- $(A_5(\alpha_1))^{op} \cong A_5\left(\frac{\alpha_1}{3\alpha_1-1}\right)$, *whenever* $\alpha_1 \neq 1/3$, $\left(A_5\left(\frac{1}{3}\right)\right)^{op} = A_9$,
- $(A_6(\alpha_1, \beta_1))^{op} = A_2(\alpha_1, \beta_1, -\alpha_1)$, $(A_7(\beta_1))^{op} = A_3(\beta_1, 0)$, $(A_8(\alpha_1))^{op} = A_4(\alpha_1, -\alpha_1)$,
- $(A_9)^{op} = A_5\left(\frac{1}{3}\right)$, $(A_{10})^{op} = A_{10}$, $(A_{11})^{op} = A_{11}$, $(A_{12})^{op} = A_{12}$.

Proof. We provide non-trivial base change that brings an algebra to its opposite unless they are equal.

Indeed, $g(A_1(\alpha_1, \alpha_2, \alpha_4, \beta_1))^{op}(g^{-1} \otimes g^{-1}) = \begin{pmatrix} -\alpha_2 & -\alpha_1 & 1-\alpha_1 & \beta_1 \\ \alpha_4 & \alpha_2 & \alpha_2+1 & \alpha_1 \end{pmatrix} = A_1(-\alpha_2, -\alpha_1, \beta_1, \alpha_4)$, where $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- If $\alpha_1 + \beta_2 \neq 0$ then $g(A_2(\alpha_1, \beta_1, \beta_2))^{op}(g^{-1} \otimes g^{-1}) = A_2\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{\beta_1}{(\alpha_1+\beta_2)\sqrt{\alpha_1+\beta_2}}, \frac{1-\alpha_1}{\alpha_1+\beta_2}\right)$, where $g = \begin{pmatrix} \alpha_1+\beta_2 & 0 \\ 0 & \sqrt{\alpha_1+\beta_2} \end{pmatrix}$;
- If $\alpha_1 + \beta_2 = 0$ then $(A_2(\alpha_1, \beta_1, -\alpha_1))^{op} = A_6(\alpha_1, \beta_1)$.

- If $\beta_2 \neq 0$ then
$$g(A_3(\alpha_1,\beta_1,\beta_2))^{op}(g^{-1}\otimes g^{-1}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \frac{\beta_1}{\beta_2^2} & \frac{1}{\beta_2} & 1 & -1 \end{pmatrix} = A_3\left(\frac{\beta_1}{\beta_2^2}, \frac{1}{\beta_2}\right), \quad \text{where} \quad g = \begin{pmatrix} \beta_2 & 0 \\ 0 & 1 \end{pmatrix};$$

- $(A_3(\beta_1,0))^{op} = A_7(\beta_1)$.

- If $\alpha_1+\beta_2 \neq 0$ then
$$g(A_4(\alpha_1,\beta_2))^{op}(g^{-1}\otimes g^{-1}) = \begin{pmatrix} \frac{\alpha_1}{\alpha_1+\beta_2} & 0 & 0 & 0 \\ 0 & \frac{1-\alpha_1}{\alpha_1+\beta_2} & 1-\frac{\alpha_1}{\alpha_1+\beta_2} & 0 \end{pmatrix} = A_4\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{1-\alpha_1}{\alpha_1+\beta_2}\right),$$
where $g = \begin{pmatrix} \alpha_1+\beta_2 & 0 \\ 0 & 1 \end{pmatrix}$;

- $(A_4(\alpha_1,-\alpha_1))^{op} = A_8(\alpha_1)$.

- If $3\alpha_1 - 1 \neq 0$ and $g = \begin{pmatrix} 3\alpha_1-1 & 0 \\ 0 & (3\alpha_1-1)^2 \end{pmatrix}$ then
$$g(A_5(\alpha_1))^{op}(g^{-1}\otimes g^{-1}) = \begin{pmatrix} \frac{\alpha_1}{3\alpha_1+1} & 0 & 0 & 0 \\ 1 & 2\frac{\alpha_1}{3\alpha_1-1}-1 & 1-\frac{\alpha_1}{3\alpha_1+1} & 0 \end{pmatrix} = A_5\left(\frac{\alpha_1}{3\alpha_1-1}\right);$$

- $\left(A_5\left(\frac{1}{3}\right)\right)^{op} = A_9$.

Now due to the equality $(A^{op})^{op} = A$ the above obtained isomorphisms and equalities imply that
$$(A_6(\alpha_1,\beta_1))^{op} = A_2(\alpha_1,\beta_1,-\alpha_1),\ (A_7(\beta_1))^{op} = A_3(\beta_1,0),$$
$$(A_8(\alpha_1))^{op} = A_4(\alpha_1,-\alpha_1),\ (A_9)^{op} = A_5\left(\frac{1}{3}\right).$$

The equalities $(A_{10})^{op} = A_{10},\ (A_{11})^{op} = A_{11},\ (A_{12})^{op} = A_{12}$ are evident. □

Corollary 4.2. *Let Char*$(\mathbb{F}) \neq 2,3$. *Then, up to isomorphism, there exist only the following nontrivial two-dimensional algebras* $\mathbb{A}$ *with* $\mathbb{A}^{op} \cong \mathbb{A}$:
$$A_1(\alpha_1,-\alpha_1,\alpha_2,\alpha_2),\ A_2(\alpha_1,\beta_1,1-\alpha_1),\ A_2(0,0,-1),\ A_3(\beta_1,-1),\ A_3(\beta_1,1),\ A_4(\alpha_1,1-\alpha_1),$$
$$A_4(0,-1),\ A_5\left(\frac{2}{3}\right),\ A_5(0),\ A_{10},\ A_{11},\ A_{12}.$$

Theorem 4.3. *Let Char*$(\mathbb{F}) = 2$. *Then the following isomorphisms and equalities hold true*

- $(A_{1,2}(\alpha_1,\alpha_2,\alpha_4,\beta_1))^{op} \cong A_{1,2}(\alpha_2,\alpha_1,\beta_1,\alpha_4)$,

- $(A_{2,2}(\alpha_1,\beta_1,\beta_2))^{op} \cong A_{2,2}\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{\beta_1}{(\alpha_1+\beta_2)\sqrt{\alpha_1+\beta_2}}, \frac{1+\alpha_1}{\alpha_1+\beta_2}\right)$, *whenever* $\alpha_1+\beta_2 \neq 0$, $(A_{2,2}(\alpha_1,\beta_1,\alpha_1))^{op} = A_{6,2}(\alpha_1,\beta_1)$,

- $(A_{3,2}(\alpha_1,\beta_2))^{op} \cong A_{3,2}\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{1+\alpha_1}{\alpha_1+\beta_2}\right)$, *whenever* $\alpha_1+\beta_2 \neq 0$, $(A_{3,2}(\alpha_1,\alpha_1))^{op} = A_{7,2}(\alpha_1)$,

- $(A_{4,2}(\alpha_1,\beta_2))^{op} \cong A_{4,2}\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{1+\alpha_1}{\alpha_1+\beta_2}\right)$, *whenever* $\alpha_1+\beta_2 \neq 0$, $(A_{4,2}(\alpha_1,\alpha_1))^{op} = A_{8,2}(\alpha_1)$,
- $(A_{5,2}(\alpha_1))^{op} \cong A_{5,2}\left(\frac{\alpha_1}{\alpha_1+1}\right)$, *whenever* $\alpha_1 \neq 1$, $(A_{5,2}(1))^{op} = A_{9,2}$,
- $(A_{6,2}(\alpha_1,\beta_1))^{op} = A_{2,2}(\alpha_1,\beta_1,\alpha_1)$, $(A_{7,2}(\alpha_1))^{op} = A_{3,2}(\alpha_1,\alpha_1)$, $(A_{8,2}(\alpha_1))^{op} = A_{4,2}(\alpha_1,\alpha_1)$,
- $(A_{9,2})^{op} = A_{5,2}(1)$, $(A_{10,2})^{op} = A_{10,2}$, $(A_{11,2})^{op} = A_{11,2}$, $(A_{12,2})^{op} = A_{12,2}$.

Proof. The proof is similar to that of previous theorem, for example,

$$g(A_{3,2}(\alpha_1,\beta_2))^{op}(g^{-1})^{\otimes 2} = A_{3,2}\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{1+\alpha_1}{\alpha_1+\beta_2}\right), \quad g = \begin{pmatrix}\alpha_1+\beta_2 & 0\\ 0 & 1\end{pmatrix}, \quad \alpha_1+\beta_2 \neq 0.$$

□

Corollary 4.4. *Let Char*$(\mathbb{F}) = 2$. *Then, up to isomorphism, there exist only the following nontrivial two-dimensional algebras with* $\mathbb{A}^{op} \cong \mathbb{A}$:
$A_{1,2}(\alpha_1,\alpha_1,\alpha_2,\alpha_2)$, $A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{5,2}(0)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

Theorem 4.5. *Let Char*$(\mathbb{F}) = 3$. *The following isomorphisms and equalities are true*

- $(A_{1,3}(\alpha_1,\alpha_2,\alpha_4,\beta_1))^{op} \cong A_{1,3}(-\alpha_2,-\alpha_1,\beta_1,\alpha_4)$,
- $(A_{2,3}(\alpha_1,\beta_1,\beta_2))^{op} \cong A_{2,3}\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{\beta_1}{(\alpha_1+\beta_2)\sqrt{\alpha_1+\beta_2}}, \frac{1-\alpha_1}{\alpha_1+\beta_2}\right)$, *whenever* $\alpha_1+\beta_2 \neq 0$, $(A_{2,3}(\alpha_1,\beta_1,-\alpha_1))^{op} = A_{6,3}(\alpha_1,\beta_1)$,
- $(A_{3,3}(\beta_1,\beta_2))^{op} \cong A_{3,3}\left(\frac{\beta_1}{\beta_2^2}, \frac{1}{\beta_2}\right)$, *whenever* $\beta_2 \neq 0$, $(A_{3,3}(\beta_1,0))^{op} = A_{7,3}(\beta_1)$,
- $(A_{4,3}(\alpha_1,\beta_2))^{op} \cong A_{4,3}\left(\frac{\alpha_1}{\alpha_1+\beta_2}, \frac{1-\alpha_1}{\alpha_1+\beta_2}\right)$, *whenever* $\alpha_1+\beta_2 \neq 0$, $(A_{4,3}(\alpha_1,-\alpha_1))^{op} = A_{8,3}(\alpha_1)$,
- $(A_{5,3}(\alpha_1))^{op} \cong A_{5,3}(-\alpha_1)$,
- $(A_{6,3}(\alpha_1,\beta_1))^{op} = A_{2,3}(\alpha_1,\beta_1,-\alpha_1)$, $(A_{7,3}(\beta_1))^{op} = A_{3,3}(\beta_1,0)$, $(A_{8,3}(\alpha_1))^{op} = A_{4,3}(\alpha_1,-\alpha_1)$,
- $(A_{9,3})^{op} = A_{9,3}$, $(A_{10,3})^{op} = A_{10,3}$, $(A_{11,3})^{op} = A_{11,3}$, $(A_{12,3})^{op} = A_{12,3}$.

Corollary 4.6. *In the case of Char*$(\mathbb{F}) = 3$, *up to isomorphism, there exist only the following nontrivial two-dimensional algebras with* $\mathbb{A}^{op} \cong \mathbb{A}$:

$$A_{1,3}(\alpha_1,-\alpha_1,\alpha_2,\alpha_2),\ A_{2,3}(\alpha_1,\beta_1,1-\alpha_1),\ A_{2,3}(0,0,-1),\ A_{3,3}(\beta_1,-1),\ A_{3,3}(\beta_1,1),$$

$$A_{4,3}(\alpha_1,1-\alpha_1),\ A_{4,3}(0,-1),\ A_{5,3}(0),\ A_{9,3},\ A_{10,3},\ A_{11,3},\ A_{12,3}.$$

Now we consider a set of the most important identities and classify all nontrivial two-dimensional algebras with respect to this set of identities. The set of identities mainly consists of identities which have appeared before in definitions of different classes of algebras and their "anti" variants. We tried not include both of an identity and its "opposite" versions into this set except for the case when an identity and its opposite are the same. Therefore, in calling identities the term "Left" is used. Of course, the use of terms "Left" and "Anti" seems to be a questionable, but we have tried to follow the tradition given in the literature earlier. From this classification one can easily derive the classification of two-dimensional algebras with a given subset of identities and due to Theorems 4.1, 4.3 and 4.5, the classification of two-dimensional algebras satisfying the corresponding "Right" ("Opposite") identities also can easily be obtained.

In the theorems below we give polynomial identities denoted by $I_1, I_2, ..., I_{30}$ followed by the classification of two-dimensional algebras satisfying these identities in terms of canonical representatives $A_1 - A_{12}$.

The following notations are used: I stands for the second order identity matrix, i is a fixed element of $\mathbb{F}$ such that $i^2 = -1$. Elements of $\mathbb{A}$ and their position vectors are denoted by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and u, v, w, respectively.

Theorem 4.7. *Let $Char(\mathbb{F}) \neq 2$, 3. The following classification of two-dimensional algebras with respect to $I_1 - I_{30}$ holds true:*

I_1. *Commutativity identity* $\mathbf{uv} = \mathbf{vu}$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_2. *Anti-commutativity identity* $\mathbf{uv} = -\mathbf{vu}$. $A_4(0, -1)$.

I_3. *Associativity identity* $(\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw})$.
$A_2\left(\frac{1}{2}, 0, \frac{1}{2}\right)$, $A_4\left(\frac{1}{2}, \frac{1}{2}\right)$, $A_4(1, 0)$, $A_4(1, 1)$, $A_4\left(\frac{1}{2}, 0\right)$, A_{12}.

I_4. *Anti-associativity identity* $(\mathbf{uv})\mathbf{w} = -\mathbf{u}(\mathbf{vw})$. A_{12}.

I_5. *Well defined cube identity* $\mathbf{u}^2\mathbf{u} = \mathbf{u}\mathbf{u}^2$.
$A_1\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right)$, $A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, where $\alpha_1 \neq \frac{2}{3}$, $A_4(\alpha_1, 2\alpha_1 - 1)$, $A_5\left(\frac{2}{3}\right)$, $A_8\left(\frac{1}{3}\right)$, A_{10}, A_{11}, A_{12}.

I_6. *Half-commutativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}[\mathbf{u}, \mathbf{v}]$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_7. *Anti-half-commutativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = -\mathbf{w}[\mathbf{u}, \mathbf{v}]$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_4(\alpha_1, \alpha_1 - 1)$, $A_5(0)$, $A_5(\frac{2}{3})$, $A_8(\frac{1}{2})$, A_{10}, A_{11}, A_{12}.

I_8. *Mixed associativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{u}[\mathbf{v}, \mathbf{w}]$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_9. *Anti-mixed-associativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = -\mathbf{u}[\mathbf{v}, \mathbf{w}]$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_{10}. *Flexibility identity* $\mathbf{u}(\mathbf{vu}) = (\mathbf{uv})\mathbf{u}$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_4(\alpha_1, 2\alpha_1 - 1)$, where $\alpha_1 \neq \frac{2}{3}$, $A_5(\frac{2}{3})$, $A_8(\frac{1}{3})$, A_{10}, A_{11}, A_{12}.

I_{11}. *Anti-flexibility identity* $\mathbf{u}(\mathbf{vu}) = -(\mathbf{uv})\mathbf{u}$. A_{12}.

I_{12}. *Mixed flexibility identity* $\mathbf{u}[\mathbf{v}, \mathbf{u}] = [\mathbf{u}, \mathbf{v}]\mathbf{u}$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_4(\alpha_1, \alpha_1 - 1)$, $A_5(0)$, $A_5(\frac{2}{3})$, $A_8(\frac{1}{2})$, A_{10}, A_{11}, A_{12}.

I_{13}. *Mixed anti-flexibility identity* $\mathbf{u}[\mathbf{v}, \mathbf{u}] = -[\mathbf{u}, \mathbf{v}]\mathbf{u}$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_{14}. *Left Leibniz identity* $\mathbf{u}(\mathbf{vw}) = (\mathbf{uv})\mathbf{w} + \mathbf{v}(\mathbf{uw})$. $A_4(0, -1), A_8(0), A_{12}$.

I_{15}. *Left anti-Leibniz identity* $\mathbf{u}(\mathbf{vw}) = -(\mathbf{uv})\mathbf{w} - \mathbf{v}(\mathbf{uw})$. A_{12}.

I_{16}. *Mixed left Leibniz identity* $\mathbf{u}[\mathbf{v}, \mathbf{w}] = [\mathbf{u}, \mathbf{v}]\mathbf{w} + \mathbf{v}[\mathbf{u}, \mathbf{w}]$.
$A_2(\alpha_1, \beta_1, 1-\alpha_1) \cong A_2(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_3(\beta_1, 1)$, $A_4(\alpha_1, 1-\alpha_1)$, $A_4(\alpha_1, \alpha_1 - 1)$, $A_5(0)$, $A_5(\frac{2}{3})$, $A_8(\frac{1}{2})$, A_{10}, A_{11}, A_{12}.

I_{17}. *Mixed anti-left Leibniz identity* $\mathbf{u}[\mathbf{v},\mathbf{w}] = -[\mathbf{u},\mathbf{v}]\mathbf{w} - \mathbf{v}[\mathbf{u},\mathbf{w}]$.
$A_2(\alpha_1,\beta_1,1-\alpha_1) \cong A_2(\alpha_1,-\beta_1,1-\alpha_1)$, $A_3(\beta_1,1)$, $A_4(\alpha_1,1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_{18}. *Left Poisson identity* $(\mathbf{uv})\mathbf{w} + (\mathbf{vw})\mathbf{u} + (\mathbf{wu})\mathbf{v} = 0$. $A_4(0,-1)$, $A_8(0)$, A_{12}.

I_{19}. *Left Jordan identity* $(\mathbf{uv})\mathbf{u}^2 = \mathbf{u}(\mathbf{vu}^2)$.

- *If* $Char(\mathbb{F}) \neq 5$ *then* $A_2\left(\frac{1}{2},0,\frac{1}{2}\right)$, $A_2\left(\frac{1}{2},0,-\frac{1}{2}\right)$, $A_4(\alpha_1,-1+2\alpha_1)$, *where* $\alpha_1 \neq \frac{1}{10}\left(5 \pm \sqrt{5}\right)$, $A_4\left(\alpha_1,\sqrt{\alpha_1-\alpha_1^2}\right)$, $A_4\left(\alpha_1,-\sqrt{\alpha_1-\alpha_1^2}\right)$, *where* $\alpha_1 \neq 0,1$, $A_5\left(\frac{1}{10}\left(5-\sqrt{5}\right)\right)$, $A_5\left(\frac{1}{10}\left(5+\sqrt{5}\right)\right)$, $A_8\left(\frac{1}{3}\right)$, $A_8\left(\frac{1}{2}-\frac{i}{2}\right)$, $A_8\left(\frac{1}{2}+\frac{i}{2}\right)$, A_{12}.
- *If* $Char(\mathbb{F}) = 5$ *then* $A_2\left(\frac{1}{2},0,\frac{1}{2}\right)$, $A_2\left(\frac{1}{2},0,-\frac{1}{2}\right)$, $A_4(\alpha_1,-1+2\alpha_1)$, $A_4\left(\alpha_1,\sqrt{\alpha_1-\alpha_1^2}\right)$, $A_4\left(\alpha_1,-\sqrt{\alpha_1-\alpha_1^2}\right)$, *where* $\alpha_1 \neq 0,1$, $A_8\left(\frac{1}{3}\right)$, $A_8\left(\frac{3}{2}\right)$, A_9, A_{12}.

I_{20}. *Left anti-Jordan identity* $(\mathbf{uv})\mathbf{u}^2 = -\mathbf{u}(\mathbf{vu}^2)$. $A_4(0,-1)$, $A_4(0,0)$, A_{12}.

I_{21}. *Mixed left Jordan identity* $[\mathbf{u},\mathbf{v}]\mathbf{u}^2 = \mathbf{u}[\mathbf{v},\mathbf{u}^2]$.
$A_2(\alpha_1,\beta_1,1-\alpha_1) \cong A_2(\alpha_1,-\beta_1,1-\alpha_1)$, $A_3(\beta_1,1)$, $A_4(0,-1)$, $A_4(0,0)$, $A_4(\alpha_1,1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_{22}. *Mixed anti-left Jordan identity* $[\mathbf{u},\mathbf{v}]\mathbf{u}^2 = -\mathbf{u}[\mathbf{v},\mathbf{u}^2]$.
$A_2(\alpha_1,\beta_1,1-\alpha_1) \cong A_2(\alpha_1,-\beta_1,1-\alpha_1)$, $A_3(\beta_1,1)$, $A_4(0,-1)$, $A_4(0,0)$, $A_4(\alpha_1,1-\alpha_1)$, $A_5(\frac{2}{3})$, A_{10}, A_{11}, A_{12}.

I_{23}. *Left Malcev identity* $((\mathbf{uv})\mathbf{w} + (\mathbf{vw})\mathbf{u} + (\mathbf{wu})\mathbf{v})\mathbf{u} = (\mathbf{uv})(\mathbf{uw}) + (\mathbf{v}(\mathbf{uw}))\mathbf{u} + ((\mathbf{uw})\mathbf{u})\mathbf{v}$.
$A_2(\frac{1}{2},0,\frac{1}{2})$, $A_4(0,-1)$, $A_4(\frac{1}{2},\frac{1}{2})$, $A_4(1,0)$, $A_8(0)$, A_{12}.

I_{24}. *Left anti-Malcev identity* $((\mathbf{uv})\mathbf{w} + (\mathbf{vw})\mathbf{u} + (\mathbf{wu})\mathbf{v})\mathbf{u} = -(\mathbf{uv})(\mathbf{uw}) - (\mathbf{v}(\mathbf{uw}))\mathbf{u} - ((\mathbf{uw})\mathbf{u})\mathbf{v}$. $A_4(0,-1)$, $A_8(0)$, A_{12}.

I_{25}. *Left Zinbiel identity* $(\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw} + \mathbf{wv})$. A_{12}.

I_{26}. *Left anti-Zinbiel identity* $(\mathbf{uv})\mathbf{w} = -\mathbf{u}(\mathbf{vw} + \mathbf{wv})$. A_{12}.

I_{27}. *Left symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = [\mathbf{v},\mathbf{u},\mathbf{w}]$.
$A_2(\frac{1}{2},0,\frac{1}{2})$, $A_2(1,0,\frac{1}{2})$, $A_4(1,\beta_2)$, $A_4(\frac{1}{2},\beta_2)$, $A_5(1)$, $A_5(\frac{1}{2})$, $A_8(0)$, A_{12}.

I_{28}. *Left anti-symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = -[\mathbf{v},\mathbf{u},\mathbf{w}]$.
$A_2(\frac{1}{2},0,\frac{1}{2})$, $A_4(\frac{1}{2},\frac{1}{2})$, $A_4(\frac{1}{2},0)$, $A_4(1,0)$, $A_4(1,1)$, A_{12}.

I_{29}. *Centro-symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = [\mathbf{w},\mathbf{v},\mathbf{u}]$.
$A_1(\alpha_1,\alpha_2,-\alpha_1-2\alpha_2,2\alpha_1+\alpha_2)$, *where* $\alpha_2 = \frac{1}{8}\left(\pm\sqrt{32\alpha_1-15}-8\alpha_1-1\right)$, $A_2\left(\frac{1}{2},0,\frac{1}{2}\right)$, $A_4\left(\alpha_1,\frac{\alpha_1-\sqrt{12\alpha_1-7\alpha_1^2-4}}{2}\right)$, $A_4\left(\alpha_1,\frac{\alpha_1+\sqrt{12\alpha_1-7\alpha_1^2-4}}{2}\right)$, $A_5\left(\frac{1}{2}\right)$, $A_5(1)$, $A_8\left(\frac{3-\sqrt{7}i}{8}\right)$, $A_8\left(\frac{3+\sqrt{7}i}{8}\right)$, A_{12}.

I_{30}. *Centro-anti-symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = -[\mathbf{w},\mathbf{v},\mathbf{u}]$.
$A_2(\alpha_1,\beta_1,1-\alpha_1)$, $A_3(\beta_1,1)$, $A_4(\alpha_1,1-\alpha_1)$, $A_4(\alpha_1,2\alpha_1-1)$, $A_5(\frac{2}{3})$, $A_8(\frac{1}{3})$, A_{10}, A_{11}, A_{12}.

Proof. Proofs of the cases $I_1 - I_{30}$ are similar. The basic idea behind the proof of the theorem is just to convert the given identity in terms of MSC A as a matrix equation and solve the corresponding system of polynomials equations for each of $A_1 - A_{12}$ canonical representatives. Here we provide the systems of equations. The solutions to them are a bit technical, but once we substitute the structure constants into the system it is much simplified and can be managed combining manual and computer calculations. We illustrate the technique in two specific cases (the cases I_{18} and I_{23}).

The cases of commutativity and anticommutativity are clear where the identities are equivalent to the fact that the second and third columns of A to be the same and differ in sign as vectors, respectively.

The proof of the cases I_5 and I_{19} are given in [3] and [7], respectively.

The identities I_3 and I_4 in terms of matrices are expressed by $A(A \otimes I) = A(I \otimes A)$ and $A(A \otimes I) = -A(I \otimes A)$, respectively.

- I_6 case. $A(A \otimes I)((u \otimes v - v \otimes u) \otimes w) = A(I \otimes A)(w \otimes (u \otimes v - v \otimes u))$.
- I_7 case. $A(A \otimes I)((u \otimes v - v \otimes u) \otimes w) = -A(I \otimes A)(w \otimes (u \otimes v - v \otimes u))$.
- I_9 case. $A(A \otimes I)((u \otimes v - v \otimes u) \otimes w) = -A(I \otimes A)(u \otimes (v \otimes w - w \otimes v))$.
- I_{10} case. $(A(I \otimes A) - A(A \otimes I))(u \otimes v \otimes u) = 0$.
- I_{11} case. $(A(I \otimes A) + A(A \otimes I))(u \otimes v \otimes u) = 0$.
- I_{12} case. $A(A \otimes I)((u \otimes v - v \otimes u) \otimes u) = A(I \otimes A)(u \otimes (v \otimes u - u \otimes v))$.
- I_{13} case. $A(A \otimes I)((u \otimes v - v \otimes u) \otimes u) = -A(I \otimes A)(u \otimes (v \otimes u - u \otimes v))$.
- I_{14} case. $(A(I \otimes A) - A(A \otimes I))(u \otimes v \otimes w) = A(I \otimes A)(v \otimes u \otimes w)$.
- I_{15} case. $(A(I \otimes A) + A(A \otimes I))(u \otimes v \otimes w) = -A(I \otimes A)(v \otimes u \otimes w)$.
- I_{16} case. $A(I \otimes A)(u \otimes (v \otimes w - w \otimes v)) = A(A \otimes I)((u \otimes v - v \otimes u) \otimes w) + A(I \otimes A)(v \otimes (u \otimes w - w \otimes u))$.
- I_{17} case. $A(I \otimes A)(u \otimes (v \otimes w - w \otimes v)) = -A(A \otimes I)((u \otimes v - v \otimes u) \otimes w) - A(I \otimes A)(v \otimes (u \otimes w - w \otimes u))$.
- I_{18} case. $A(A \otimes I)(u \otimes v \otimes w + v \otimes w \otimes u + w \otimes u \otimes v) = 0$.
- I_{20} case. $(A(A \otimes A) + A(I \otimes A(I \otimes A)))(u \otimes v \otimes u \otimes u) = 0$.
- I_{21} case. $A(A \otimes A)((u \otimes v - v \otimes u) \otimes u^{\otimes 2}) = A(I \otimes A(I \otimes A))(u \otimes v \otimes u^{\otimes 2}) - A(I \otimes A(A \otimes I))(u^{\otimes 3} \otimes v)$.
- I_{22} case. $A(A \otimes A)((u \otimes v - v \otimes u) \otimes u^{\otimes 2}) = -A(I \otimes A(I \otimes A))(u \otimes v \otimes u^{\otimes 2}) + A(I \otimes A(A \otimes I))(u^{\otimes 3} \otimes v)$.
- I_{23} case. $A(A(A \otimes I) \otimes I)((u \otimes v \otimes w + v \otimes w \otimes u + w \otimes u \otimes v) \otimes u) = A(A \otimes A)(u \otimes v \otimes u \otimes w) + A(A(I \otimes A) \otimes I)(v \otimes u \otimes w \otimes u) + A(A(A \otimes I) \otimes I)(u \otimes w \otimes u \otimes v)$.
- I_{24} case. $A(A(A \otimes I) \otimes I)((u \otimes v \otimes w + v \otimes w \otimes u + w \otimes u \otimes v) \otimes u) = -A(A \otimes A)(u \otimes v \otimes u \otimes w) - A(A(I \otimes A) \otimes I)(v \otimes u \otimes w \otimes u) - A(A(A \otimes I) \otimes I)(u \otimes w \otimes u \otimes v)$.
- I_{25} case. $A(A \otimes I)(u \otimes v \otimes w) = A(I \otimes A)(u \otimes v \otimes w) + A(I \otimes A)(u \otimes w \otimes v)$.
- I_{26} case. $A(A \otimes I)(u \otimes v \otimes w) = -A(I \otimes A)(u \otimes v \otimes w) - A(I \otimes A)(u \otimes w \otimes v)$.
- I_{27} case. $A((A \otimes I) - (I \otimes A))(u \otimes v \otimes w) = A((A \otimes I) - (I \otimes A))(v \otimes u \otimes w)$.

- I_{28} case. $A((A\otimes I)-(I\otimes A))(u\otimes v\otimes w)=-A((A\otimes I)-(I\otimes A))(v\otimes u\otimes w)$.
- I_{29} case. $A((A\otimes I)-(I\otimes A))(u\otimes v\otimes w)=A((A\otimes I)-(I\otimes A))(w\otimes v\otimes u)$.
- I_{30} case. $A((A\otimes I)-(I\otimes A))(u\otimes v\otimes w)=-A((A\otimes I)-(I\otimes A))(w\otimes v\otimes u)$.

□

Remark. *Note that in the case of* I_{19} *if* $Char(\mathbb{F})=5$ *then* $i=2$ *and thus here we correct an inaccuracy admitted in Theorem 7 from* [7].

The analogues of the result above for the fields of characteristic two and three can be easily proved following the same manner. Here we give final results without proof as the following two theorems below. In the case of the characteristic two some of the identities coincide, this is also denoted by $\cong$.

Theorem 4.9. *Let* $Char(\mathbb{F})=2$. *Then the following classification of two-dimensional algebras over* $\mathbb{F}$ *with respect to the identities* I_1-I_{30} *is valid.*

[$I_1\cong I_2$.] *Commutativity identity* $\mathbf{uv}=\mathbf{vu}$.
$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{5,2}(0)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[$I_3\cong I_4$.] *Associativity identity*
$A_{3,2}(1,0)$, $A_{4,2}(1,0)$, $A_{4,2}(1,1)$, $A_{8,2}(1)$, $A_{10,2}$, $A_{12,2}$.

[I_5.] *Well defined cube identity* $\mathbf{u}^2\mathbf{u}=\mathbf{u}\mathbf{u}^2$.
$A_{1,2}(1,1,0,0)$, $A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, *where* $\alpha_1\neq 0$, $A_{4,2}(\alpha_1,1)$, $A_{5,2}(0)$, $A_{8,2}(1)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[$I_6\cong I_7$.] *Half-commutativity identity* $[\mathbf{u},\mathbf{v}]\mathbf{w}=\mathbf{w}[\mathbf{u},\mathbf{v}]$.
$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{5,2}(0)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[$I_8\cong I_9$.] *Mixed associativity identity* $[\mathbf{u},\mathbf{v}]\mathbf{w}=\mathbf{u}[\mathbf{v},\mathbf{w}]$.
$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{5,2}(0)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[$I_{10}\cong I_{11}$.] *Flexibility identity* $\mathbf{u}(\mathbf{vu})=(\mathbf{uv})\mathbf{u}$.
$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1)$, *where* $\alpha_1\neq 0$, $A_{5,2}(0)$, $A_{8,2}(1)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[$I_{12}\cong I_{13}$.] *Mixed flexibility identity* $\mathbf{u}[\mathbf{v},\mathbf{u}]=[\mathbf{u},\mathbf{v}]\mathbf{u}$.
$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{5,2}(0)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[$I_{14}\cong I_{15}$.] *Left Leibniz identity* $\mathbf{u}(\mathbf{vw})=(\mathbf{uv})\mathbf{w}+\mathbf{v}(\mathbf{uw})$.
$A_{4,2}(0,1)$, $A_{8,2}(0)$, $A_{12,2}$.

[$I_{16}\cong I_{17}$.] *Mixed left Leibniz identity* $\mathbf{u}[\mathbf{v},\mathbf{w}]=[\mathbf{u},\mathbf{v}]\mathbf{w}+\mathbf{v}[\mathbf{u},\mathbf{w}]$.
$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{5,2}(0)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[I_{18}.] *Left Poisson identity* $(\mathbf{uv})\mathbf{w}+(\mathbf{vw})\mathbf{u}+(\mathbf{wu})\mathbf{v}=0$.
$A_{4,2}(0,1)$, $A_{5,2}(0)$, $A_{8,2}(0)$, $A_{12,2}$.

[$I_{19}\cong I_{20}$.] *Left Jordan identity* $(\mathbf{uv})\mathbf{u}^2=\mathbf{u}(\mathbf{vu}^2)$.
$A_{3,2}(0,0)$, $A_{3,2}(1,0)$, $A_{4,2}(\alpha_1,1)$, $A_{4,2}\left(\alpha_1,\sqrt{\alpha_1+\alpha_1^2}\right)$, *where* $\alpha_1^2+\alpha_1+1\neq 0$, $A_{5,2}(\alpha_1)$, *where* $\alpha_1^2+\alpha_1+1=0$, $A_{8,2}(1)$, $A_{10,2}$, $A_{12,2}$.

[$I_{21}\cong I_{22}$.] *Mixed left Jordan identity* $[\mathbf{u},\mathbf{v}]\mathbf{u}^2=\mathbf{u}[\mathbf{v},\mathbf{u}^2]$.
$A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(0,0)$, $A_{5,2}(0)$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

[$I_{23}\cong I_{24}$.] *Left Malcev identity* $((\mathbf{uv})\mathbf{w}+(\mathbf{vw})\mathbf{u}+(\mathbf{wu})\mathbf{v})\mathbf{u}=(\mathbf{uv})(\mathbf{uw})+(\mathbf{v}(\mathbf{uw}))\mathbf{u}+((\mathbf{uw})\mathbf{u})\mathbf{v}$.
$A_{3,2}(1,0)$, $A_{4,2}(1,0)$, $A_{4,2}(0,1)$, $A_{8,2}(0)$, $A_{10,2}$, $A_{12,2}$.

[$I_{25}\cong I_{26}$.] *Left Zinbiel identity* $(\mathbf{uv})\mathbf{w}=\mathbf{u}(\mathbf{vw}+\mathbf{wv})$. $A_{12,2}$.

[$I_{27}\cong I_{28}$.] *Left symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}]=[\mathbf{v},\mathbf{u},\mathbf{w}]$.
$A_{3,2}(1,0)$, $A_{4,2}(1,\beta_2)$, $A_{5,2}(1)$, $A_{6,2}(\alpha_1,0)$, $A_{7,2}(1)$, $A_{8,2}(\alpha_1)$, $A_{9,2}$, $A_{10,2}$, $A_{12,2}$.

[$I_{29}\cong I_{30}$.] *Centro-symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}]=[\mathbf{w},\mathbf{v},\mathbf{u}]$.
$A_{1,2}(\alpha_1,1+\alpha_1,\alpha_1,1+\alpha_1)$, $A_{2,2}(\alpha_1,\beta_1,1+\alpha_1)$, $A_{3,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1+\alpha_1)$, $A_{4,2}(\alpha_1,1)$, where $\alpha_1\neq 0$, $A_{5,2}(\alpha_1)$, $A_{8,2}(1)$, $A_{9,2}$, $A_{10,2}$, $A_{11,2}$, $A_{12,2}$.

Theorem 4.10. *In the case of* $Char(\mathbb{F})=3$ *the following classification result of two-dimensional algebras over* $\mathbb{F}$ *with respect to the identities* I_1-I_{30} *holds true.*

I_1. *Commutativity identity* $\mathbf{uv} = \mathbf{vu}$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{5,3}(\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_2. *Anti-commutativity identity* $\mathbf{uv} = -\mathbf{vu}$. $A_4(0, -1)$.

I_3. *Associativity identity*
$A_{2,3}\,(-1, 0, -1)$, $A_{4,3}\,(-1, -1)$, $A_{4,3}(1, 0)$, $A_{4,3}\,(1, 1)$, $A_{4,3}\,(-1, 0)$, $A_{12,3}$.

I_4. *Anti-associativity identity* $(\mathbf{uv})\mathbf{w} = -\mathbf{u}(\mathbf{vw})$, $A_{12,3}$.

I_5. *Well defined cube identity*
$A_{2,3}\,(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}\,(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}\,(\beta_1, 1)$, $A_{3,3}\,(0, -1)$, $A_{4,3}\,(\alpha_1, 1-\alpha_1)$, $A_{4,3}\,(\alpha_1, 2\alpha_1 - 1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_6. *Half-commutativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}[\mathbf{u}, \mathbf{v}]$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_7. *Anti-half-commutativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = -\mathbf{w}[\mathbf{u}, \mathbf{v}]$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{4,3}(\alpha_1, \alpha_1 - 1)$, $A_{5,3}(0)$, $A_{8,3}(-1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_8. *Mixed associativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{u}[\mathbf{v}, \mathbf{w}]$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_9. *Anti-mixed-associativity identity* $[\mathbf{u}, \mathbf{v}]\mathbf{w} = -\mathbf{u}[\mathbf{v}, \mathbf{w}]$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_{10}. *Flexibility identity* $\mathbf{u}(\mathbf{vu}) = (\mathbf{uv})\mathbf{u}$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{4,3}(\alpha_1, 2\alpha_1 - 1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_{11}. *Anti-flexibility identity* $\mathbf{u}(\mathbf{vu}) = -(\mathbf{uv})\mathbf{u}$. $A_{12,3}$.

I_{12}. *Mixed flexibility identity* $\mathbf{u}[\mathbf{v}, \mathbf{u}] = [\mathbf{u}, \mathbf{v}]\mathbf{u}$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{4,3}(\alpha_1, \alpha_1 - 1)$, $A_{5,3}(0)$,
$A_{8,3}(-1), A_{9,3}, A_{10,3}, A_{11,3}, A_{12,3}$.

I_{13}. *Mixed anti-flexibility identity* $\mathbf{u}[\mathbf{v}, \mathbf{u}] = -[\mathbf{u}, \mathbf{v}]\mathbf{u}$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_{14}. *Left Leibniz identity* $\mathbf{u}(\mathbf{vw}) = (\mathbf{uv})\mathbf{w} + \mathbf{v}(\mathbf{uw})$.
$A_{4,3}(0, -1)$, $A_{8,3}(0)$, $A_{12,3}$.

I_{15}. *Left anti-Leibniz identity* $\mathbf{u}(\mathbf{vw}) = -(\mathbf{uv})\mathbf{w} - \mathbf{v}(\mathbf{uw})$.
$A_{2,3}(2, 0, -1)$, $A_{4,3}(1, 0)$, $A_{4,3}(1, 1)$, $A_{4,3}(-1, -1)$, $A_{12,3}$.

I_{16}. *Mixed left Leibniz identity* $\mathbf{u}[\mathbf{v}, \mathbf{w}] = [\mathbf{u}, \mathbf{v}]\mathbf{w} + \mathbf{v}[\mathbf{u}, \mathbf{w}]$.
$A_{2,3}(\alpha_1, \beta_1, 1-\alpha_1) \cong A_{2,3}(\alpha_1, -\beta_1, 1-\alpha_1)$, $A_{3,3}(\beta_1, 1)$, $A_{4,3}(\alpha_1, 1-\alpha_1)$, $A_{4,3}(\alpha_1, \alpha_1 - 1)$, $A_{5,3}(0)$, $A_{8,3}(-1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_{17}. *Mixed anti-left Leibniz identity* $\mathbf{u}[\mathbf{v},\mathbf{w}] = -[\mathbf{u},\mathbf{v}]\mathbf{w} - \mathbf{v}[\mathbf{u},\mathbf{w}]$.
$A_{2,3}(\alpha_1,\beta_1,1-\alpha_1) \cong A_{2,3}(\alpha_1,-\beta_1,1-\alpha_1)$, $A_{3,3}(\beta_1,1)$, $A_{4,3}(\alpha_1,1-\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_{18}. *Left Poisson identity* $(\mathbf{uv})\mathbf{w} + (\mathbf{vw})\mathbf{u} + (\mathbf{wu})\mathbf{v} = 0$.
$A_{2,3}(0,0,-1)$, $A_{2,3}(2,0,-1)$, $A_{4,3}(\alpha_1,-(1-\alpha_1)^2)$, $A_{5,3}(0)$, $A_{8,3}(0)$, $A_{12,3}$.

I_{19}. *Left Jordan identity* $(\mathbf{uv})\mathbf{u}^2 = \mathbf{u}(\mathbf{vu}^2)$.
$A_{2,3}(-1,0,1)$, $A_{2,3}(-1,0,-1)$, $A_{4,3}(\alpha_1,-1-\alpha_1)$, where $\alpha_1 \neq -1 \pm i$, $A_{4,3}\left(\alpha_1,\sqrt{\alpha_1-\alpha_1^2}\right)$, $A_{4,3}\left(\alpha_1,-\sqrt{\alpha_1-\alpha_1^2}\right)$, where $\alpha_1 \neq 0,1$, $A_{5,3}(-1+i)$, $A_{5,3}(-1-i)$, $A_{8,3}(-1+i)$, $A_{8,3}(-1-i)$, $A_{10,3}$, $A_{12,3}$.

I_{20}. *Left anti-Jordan identity* $(\mathbf{uv})\mathbf{u}^2 = -\mathbf{u}(\mathbf{vu}^2)$.
$A_{4,3}(0,-1)$, $A_{4,3}(0,0)$, $A_{12,3}$.

I_{21}. *Mixed left Jordan identity* $[\mathbf{u},\mathbf{v}]\mathbf{u}^2 = \mathbf{u}[\mathbf{v},\mathbf{u}^2]$.
$A_{2,3}(\alpha_1,\beta_1,1-\alpha_1) \cong A_{2,3}(\alpha_1,-\beta_1,1-\alpha_1)$, $A_{3,3}(\beta_1,1)$, $A_{4,3}(0,-1)$, $A_{4,3}(0,0)$, $A_{4,3}(\alpha_1,1-\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_{22}. *Mixed anti-left Jordan identity* $[\mathbf{u},\mathbf{v}]\mathbf{u}^2 = -\mathbf{u}[\mathbf{v},\mathbf{u}^2]$.
$A_{2,3}(\alpha_1,\beta_1,1-\alpha_1) \cong A_{2,3}2(\alpha_1,-\beta_1,1-\alpha_1)$, $A_{3,3}(\beta_1,1)$, $A_{4,3}(0,-1)$, $A_{4,3}(0,0)$, $A_{4,3}(\alpha_1,1-\alpha_1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

I_{23}. *Left Malcev identity* $((\mathbf{uv})\mathbf{w} + (\mathbf{vw})\mathbf{u} + (\mathbf{wu})\mathbf{v})\mathbf{u} = (\mathbf{uv})(\mathbf{uw}) + (\mathbf{v}(\mathbf{uw}))\mathbf{u} + ((\mathbf{uw})\mathbf{u})\mathbf{v}$.
$A_{2,3}(-1,0,-1)$, $A_{2,3}(0,0,-1)$, $A_{4,3}(\alpha_1,-(1-\alpha_1)^2)$, $A_{5,3}(0)$, $A_{8,3}(0)$, $A_{12,3}$.

I_{24}. *Left anti-Malcev identity* $((\mathbf{uv})\mathbf{w} + (\mathbf{vw})\mathbf{u} + (\mathbf{wu})\mathbf{v})\mathbf{u} = -(\mathbf{uv})(\mathbf{uw}) - (\mathbf{v}(\mathbf{uw}))\mathbf{u} - ((\mathbf{uw})\mathbf{u})\mathbf{v}$.
$A_{2,3}(-1,0,-1)$, $A_{2,3}(0,0,-1)$, $A_{4,3}(\alpha_1,-(1-\alpha_1)^2)$, $A_{5,3}(0)$, $A_{8,3}(0)$, $A_{12,3}$.

I_{25}. *Left Zinbiel identity* $(\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw} + \mathbf{wv})$.
$A_{12,3}$.

I_{26}. *Left anti-Zinbiel identity* $(\mathbf{uv})\mathbf{w} = -\mathbf{u}(\mathbf{vw} + \mathbf{wv})$.
$A_{2,3}(-1,0,-1)$, $A_{4,3}(-1,0)$, $A_{4,3}(-1,-1)$, $A_{4,3}(1,0)$, $A_{12,3}$.

I_{27}. *Left symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = [\mathbf{v},\mathbf{u},\mathbf{w}]$.
$A_{2,3}(-1,0,-1)$, $A_{2.3}(1,0,-1)$, $A_{4,3}(1,\beta_2)$, $A_{4,3}(-1,\beta_2)$, $A_{5,3}(-1)$, $A_{5,3}(1)$, $A_{8,3}(0)$, $A_{12,3}$.

I_{28}. *Left anti-symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = -[\mathbf{v},\mathbf{u},\mathbf{w}]$.
$A_{2,3}(-1,0,-1)$, $A_{4,3}(-1,-1)$, $A_{4,3}(-1,0)$, $A_{4,3}(1,0)$, $A_{4,3}(1,1)$, $A_{12,3}$.

I_{29}. *Centro-symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = [\mathbf{w},\mathbf{v},\mathbf{u}]$.
$A_{1,3}(\alpha_1,\alpha_2,-\alpha_1-2\alpha_2,2\alpha_1+\alpha_2)$, *where* $\alpha_2 = \left(\pm\sqrt{2\alpha_1}-\alpha_1+1\right)$, $A_{2,3}(-1,0,-1)$, $A_{4,3}(\alpha_1,-\alpha_1+\sqrt{-\alpha_1^2-1})$, $A_{4,3}(\alpha_1,-\alpha_1-\sqrt{-\alpha_1^2-1})$, $A_{5,3}(-1)$, $A_5(1)$, $A_{8,3}(i)$, $A_{8,3}(-i)$, $A_{12,3}$.

I_{30}. *Centro-anti-symmetric identity* $[\mathbf{u},\mathbf{v},\mathbf{w}] = -[\mathbf{w},\mathbf{v},\mathbf{u}]$.
$A_{2,3}(\alpha_1,\beta_1,1-\alpha_1) \cong A_{2,3}(\alpha_1,-\beta_1,1-\alpha_1)$, $A_{3,3}(\beta_1,1)$, $A_{4,3}(\alpha_1,1-\alpha_1)$, $A_{4,3}(\alpha_1,2\alpha_1-1)$, $A_{9,3}$, $A_{10,3}$, $A_{11,3}$, $A_{12,3}$.

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